Enumeration of bipartite graphs and bipartite blocks

Andrew Gainer-Dewar
Carleton College
againerdewar@carleton.edu

Ira M. Gessel*
Brandeis University
gessel@brandeis.edu

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Abstract

Using the theory of combinatorial species, we compute the cycle index for bipartite graphs, which we use to count unlabeled bipartite graphs and bipartite blocks.

1. Introduction

A *bicolored graph* is a graph each vertex of which has been assigned one of two colors such that each edge connects vertices of different colors. A *bipartite graph* is a graph G which admits such a coloring. Given j white and k black vertices, there are 2^{jk} ways to join vertices of different colors. Thus the number of (labeled) bicolored graphs on n vertices is

$$b_n = \sum_{i+j=n} \binom{n}{i} 2^{ij}. \tag{1}$$

Bipartite graphs are not so easy to count directly. Every *connected* bicolored graph has exactly two colorings in white and black, so we can count bipartite graphs by relating them to connected bipartite graphs. To do this, we use the exponential formula [15, section 5.1], which implies that if $f(x) = \sum_{n=1}^{\infty} f_n x^n/n!$ is the exponential generating function for a class of connected graphs then $e^{f(x)}$ is the exponential generating function for graphs all of whose connected components belong to this class. It follows that with $B(x) = \sum_{n=0}^{\infty} b_n x^n/n!$, where b_n is given by equation (1), the exponential generating function for connected bicolored graphs is $\log B(x)$, the exponential generating function for connected bipartite graphs is $\frac{1}{2}\log B(x)$, and the exponential generating function for bipartite graphs is $e^{\log B(x)/2} = \sqrt{B(x)}$.

Just as arbitrary graphs may be decomposed into their connected components, arbitrary connected graphs may be decomposed into "blocks"—maximal 2-connected (or "nonseparable") subgraphs. Techniques developed by Robinson [13] were applied by Harary and Robinson [9] to show that the exponential generating function $N(x)$ for labeled 2-connected bipartite graphs is related to the exponential generating function $P(x) = \frac{1}{2}\log B(x)$ for connected bipartite graphs by the equation $\log P'(x) = N'(xP'(x))$. This equation suffices to compute the number of labeled bipartite blocks on n vertices and their asymptotics.

To count unlabeled bipartite graphs we can take a similar approach. It is not too difficult to find the generating function for bicolored graphs from first principles, in a way that is very similar to counting unlabeled graphs (see, e.g., [7]). There is an analogue of the exponential formula for unlabeled graphs (see, e.g., [6, equation (3.1.1)] and [1, p. 46, equation (20b) and p. 55, equation (60 ii)] so we can easily relate the generating function for bicolored graphs to that for connected bicolored graphs and the generating function for connected bipartite graphs to that for all bipartite graphs. The difficult step is relating connected bicolored graphs to connected bipartite graphs: some unlabeled connected bipartite graphs can be bicolored in two different ways, and some in only one way, as shown in Fig. 1.

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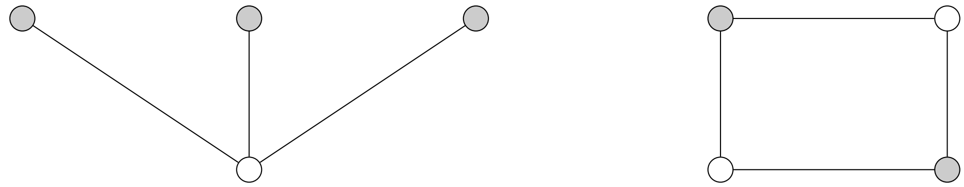

(a) A connected bipartite graph with two distinct bicolorings, one of which is shown

(b) A connected bipartite graph with just one distinct bicoloring

Figure 1: Some connected unlabeled bipartite graphs have two distinct bicolorings, but some have only one

To deal with this problem, we consider the two-element group $\mathfrak{S}_2$ acting on bicolored graphs by interchanging the colors. We want to count orbits of this group acting on connected bicolored graphs. To do this using Burnside's lemma, we would need to know the number of connected bicolored graphs fixed by each of the elements of $\mathfrak{S}_2$. This is not so easy to compute directly; however, it is not hard to compute the corresponding information for the action of $\mathfrak{S}_2$ on all bicolored graphs, and by using an extension of the exponential formula we can transfer this information to connected bicolored graphs.

Rather than just counting unlabeled bipartite graphs, which were counted earlier by Harary and Prins [8] and by Hanlon [5], we compute their cycle index, a power series in infinitely many variables that generalizes both the labeled and unlabeled enumeration, and allows us to count unlabeled bipartite blocks, which has not been previously accomplished.

2. The theory of species

2.1. Introduction

In [11], André Joyal introduced the notion of "species of structures", which place the classical idea of a "combinatorial class" (e.g., trees or permutations) in a categorical setting. A *species* is a functor from the category **FinBij** of finite sets with bijections to the category **FinSet** of finite sets with set maps[1]. We write $F[A]$ for the image of the set A under the species F; for example, if F is the species of bipartite graphs then $F[A]$ is the set of bipartite graphs whose vertices are the elements of the set A. We refer the reader to [1, §1.2] for details; we give here only a brief summary of the facts that we will need from the theory of species.

The crux of species-theoretic combinatorial analysis is to focus on "labeled" combinatorial objects while paying careful attention to the effects of maps acting on the label set—in particular, the actions of the permutation groups $\mathfrak{S}_A$ acting on each set A. (We will write $\mathfrak{S}_n$ to denote the symmetric group acting on the set $[n] := 1, 2, \ldots, n$, since in general only the cardinality of the label set will matter.) Treating a species as a functor **FinBij** $\to$ **FinSet** rather than simply as a function on the class of finite sets allows us to keep track of this permutation information. We note that the transport $F[\sigma]$ of a permutation $\sigma \in \mathfrak{S}_A$ under a species F is a permutation of the set $F[A]$ of F-structures labeled by A. An "unlabeled" F-structure on A may be understood as an orbit of F-structures under the action of $\mathfrak{S}_A$.

Classical enumerative methods frequently use the algebra of generating functions, which record the number of structures of a given size in a given class as the coefficients of a formal power series. To achieve the same goal in species-theoretic analysis, we define an analogous algebraic object which records information related to the action of the permutation groups. This object is "cycle index" of the species, a symmetric function defined as a sum in the power sum basis $p_i = \sum_j x_j^i$. (In some accounts of the theory the p_i are taken simply as independent indeterminates.)

Definition 2.1. For a species F, define its *cycle index series* to be the symmetric function

$$Z_F(p_1, p_2, \ldots) := \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} \operatorname{fix}(F[\sigma]) p_1^{\sigma_1} p_2^{\sigma_2} \ldots \right) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} \operatorname{fix}(F[\sigma]) p_\sigma \right) \tag{2}$$

where $\operatorname{fix}(F[\sigma]) := |\{s \in F[n] : F[\sigma](s) = s\}|$, σ_i is the number of i-cycles of σ, and $p_\sigma = p_1^{\sigma_1} p_2^{\sigma_2} \ldots$.

[1] The use of **FinSet** instead of **FinBij** for the target category is necessary for technical reasons related to quotients.

It is easily verified that $\mathrm{fix}(F[\sigma])$ depends only on the cycle type of σ. The cycle types of permutations $\sigma \in \mathfrak{S}_n$ are in natural bijective correspondence with integer partitions $\lambda \vdash n$ (that is, weakly decreasing sequences $(\lambda_1, \lambda_2, \ldots)$ such that $\sum_i \lambda_i = n$), and the number of permutations in $\mathfrak{S}_n$ of cycle type λ is $n!/z_\lambda$, where if λ has l_i parts equal to i for each i, then z_λ is $1^{l_1} l_1! \, 2^{l_2} l_2! \cdots$. Thus we may write the sum over permutations in equation (2) as a sum over partitions:

$$Z_F(p_1, p_2, \ldots) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \mathrm{fix}(F[\lambda]) \frac{p_\lambda}{z_\lambda}, \tag{3}$$

where $\mathrm{fix}\, F[\lambda] = \mathrm{fix}\, F[\sigma]$ and $p_\lambda = p_\sigma$ for any permutation σ of cycle type λ.

The cycle index series Z_F of a species F captures enough of its structure that we may recover both labeled and unlabeled enumerations.

Theorem 2.2. *The ordinary generating function $\tilde{F}(x)$ for unlabeled F-structures is given by*

$$\tilde{F}(x) = Z_F(x, x^2, x^3, \ldots). \tag{4}$$

A proof may be found in [1, §1.2].

Moreover, the algebra of cycle indices directly mirrors the combinatorial calculus of species. Addition, multiplication, and composition of species have natural combinatorial interpretations and correspond directly to addition, multiplication, and plethystic composition of their associated cycle indices. This last result is of particular interest:

Definition 2.3. For two species F and G with $G[\varnothing] = \varnothing$, define their *composition* to be the species $F \circ G$ given by $(F \circ G)[A] = \prod_{\pi \in P(A)} (F[\pi] \times \prod_{B \in \pi} G[B])$ where $P(A)$ is the set of partitions of A.

In other words, the composition $F \circ G$ is the species of F-structures of collections of G-structures.

Definition 2.4. Let f and g be cycle indices. Then the *plethysm* $f \circ g$ is the cycle index

$$f \circ g = f(g(p_1, p_2, p_3, \ldots), g(p_2, p_4, p_6, \ldots), \ldots), \tag{5}$$

where $f(a, b, \ldots)$ denotes the cycle index f with a substituted for p_1, b substituted for p_2, and so on.

This is the same as the definition of plethysm of symmetric functions (see, e.g., Stanley [15, p. 447]). Plethysm of cycle indices then corresponds exactly to species composition:

Theorem 2.5. *For species F and G with $G[\varnothing] = \varnothing$, the cycle index of their plethysm is*

$$Z_{F \circ G} = Z_F \circ Z_G \tag{6}$$

where $\circ$ in the right-hand side is as in equation (5)*.*

Many combinatorial structures admit natural descriptions as compositions of species. For example, every graph admits a unique decomposition as a (possibly empty) set of (nonempty) connected graphs, so we have the species identity $\mathcal{G} = \mathcal{E} \circ \mathcal{G}^C$ where $\mathcal{E}$ is the species of sets, $\mathcal{G}$ the species of graphs, and $\mathcal{G}^C$ is the species of connected graphs.

The theory of species may be extended to *virtual species*, which are formal differences of species. All of the operations for species that we have discussed extend in a straightforward way to virtual species. We refer the reader to [1, §2.5] for details. In particular, two virtual species F and G are *compositional inverses* if $F \circ G = X$ (or equivalently, $G \circ F = X$) where X is the species of singletons, defined by $X[A] = \{A\}$ if $|A| = 1$, and $X[A] = \varnothing$ otherwise. We write $F^{\langle -1 \rangle}$ for the compositional inverse of F if it exists.

2.2. Γ-species and quotient species

In classical enumerative combinatorics, Burnside's lemma (also known as the Cauchy-Frobenius lemma) is a powerful tool for incorporating group actions into enumerative computations. We now present an extension of this technique to the species-theoretic context.

Definition 2.6. For Γ a finite group, a Γ-*species* F is a combinatorial species F together with an action of Γ on F-structures which commutes with isomorphisms of those structures.

For a motivating example, consider the species $k\mathcal{CG}$ of k-colored graphs; the action of $\mathfrak{S}_k$ on the colors commutes with relabelings of graphs, so $k\mathcal{CG}$ is a $\mathfrak{S}_k$-species with respect to this action.

From a Γ-species, we can construct a quotient under the action of Γ:

Definition 2.7. For F a Γ-species, define ${}^{F}/_{\Gamma}$, the *quotient species* of F under the action of Γ, to be the species of Γ-orbits of F-structures.

A brief exposition of the notion of quotient species may be found in [1, §3.6], and a more thorough exposition in [2].

Just as with classical species, we may associate a cycle index to a Γ-species, following Henderson [10].

Definition 2.8. For a Γ-species F, we define the Γ-cycle index Z_F^Γ: for each $\gamma \in \Gamma$, let

$$Z_F^\Gamma(\gamma) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{fix}(\gamma \cdot F[\sigma]) p_\sigma \tag{7}$$

with p_σ as in equation (2).

We will call such an object (formally a map from Γ to the ring $\mathbf{Q}[[p_1, p_2, \ldots]]$ of symmetric functions with rational coefficients in the p-basis) a Γ-*cycle index* even when it is not explicitly the Γ-cycle index of a Γ-species. So the coefficients in the power series count the fixed points of the *combined* action of a permutation and the group element γ. Note that, in particular, the classical ("ordinary") cycle index may be recovered as $Z_F = Z_F^\Gamma(e)$ for any Γ-species F.

The algebraic relationships between ordinary species and their cycle indices generally extend to the Γ-species context. The actions on cycle indices of Γ-species addition and multiplication are exactly as in the ordinary species case considered Γ-componentwise. The action of composition, which in ordinary species corresponds to plethysm of cycle indices, can also be extended:

Definition 2.9. For two Γ-species F and G, define their *composition* to be the Γ-species $F \circ G$ with structures given by $(F \circ G)[A] = \prod_{\pi \in P(A)} (F[\pi] \times \prod_{B \in \pi} G[B])$ where $P(A)$ is the set of partitions of A and where $\gamma \in \Gamma$ acts on a $(F \circ G)$-structure by acting on the F-structure and the G-structures independently.

A formula similar to that of Theorem 2.5 requires a definition of the plethysm of Γ-symmetric functions, here taken from Henderson [10, §3].

Definition 2.10. For two Γ-cycle indices f and g, their *plethysm* $f \circ g$ is a Γ-cycle index defined by

$$(f \circ g)(\gamma) = f(\gamma)(g(\gamma)(p_1, p_2, p_3, \ldots), g(\gamma^2)(p_2, p_4, p_6, \ldots), \ldots). \tag{8}$$

This definition of Γ-cycle index plethysm is then indeed the correct operation to pair with the composition of Γ-species:

Theorem 2.11 (Theorem 3.1, [10])**.** *If A and B are Γ-species and $B(\varnothing) = \varnothing$, then*

$$Z_{A \circ B}^\Gamma = Z_A^\Gamma \circ Z_B^\Gamma. \tag{9}$$

Recall from equation (2) that, to compute the cycle index of a species, we need to enumerate the fixed points of each $\sigma \in \mathfrak{S}_n$. To count fixed points in the quotient species ${}^{F}/_{\Gamma}$ we need to count the fixed Γ-orbits of σ in F under commuting actions of $\mathfrak{S}_n$ and Γ (that is, under an $(\mathfrak{S}_n \times \Gamma)$-action). This may be accomplished by the following generalization of Burnside's lemma [14]. (A more general result appears in [3, Theorem 4.2b].)

Lemma 2.12. *If Γ and Δ are finite groups and S is a set with a $(\Gamma \times \Delta)$-action, then for any $\delta \in \Delta$ the number of Γ-orbits fixed by δ is $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \operatorname{fix}(\gamma, \delta)$.*

Lemma 2.12 yields a formula for the cycle index of a quotient species in terms of the Γ-cycle index.

Theorem 2.13. *For a Γ-species F, the ordinary cycle index of the quotient species ${}^{F}/_{\Gamma}$ is given by*

$$Z_{F/\Gamma} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} Z_F^{\Gamma}(\gamma) = \frac{1}{|\Gamma|} \sum_{\substack{n \geq 0 \\ \sigma \in \mathfrak{S}_n \\ \gamma \in \Gamma}} \frac{1}{n!} (\gamma \cdot F[\sigma]) p_\sigma. \tag{10}$$

We will use the notation $\overline{Z_F^{\Gamma}}$ for $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} Z_F^{\Gamma}(\gamma)$.

Note that this same result on cycle indices is implicit in [2, §2.2.3]. With it, we can compute explicit enumerative data for a quotient species using cycle-index information of the original Γ-species with respect to the group action.

We note that when Γ is a symmetric group $\mathfrak{S}_n$, as in this case, we may represent the Γ-cycle index as a symmetric function in two sets of variables which is homogeneous of degree n in the second set of variables; with this approach, Γ-cycle index plethysm corresponds to the operation of "inner plethysm in y" studied by Travis [17].

3. The species of bipartite blocks

3.1. Introduction

Definition 3.1. A *bicolored graph* is a graph G each vertex of which has been assigned one of two colors (here, black and white) such that each edge connects vertices of different colors. A *bipartite graph* (sometimes called *bicolorable*) is a graph G which admits such a coloring.

There is an extensive literature about bicolored and bipartite graphs, including enumerative results for bicolored graphs [7], bipartite graphs both allowing [5] and prohibiting [8] isolated points, and bipartite blocks [9]. However, the enumeration of bipartite blocks has been accomplished previously only in the labeled case. By considering the problem in light of the theory of Γ-species, we develop a more systematic understanding of the structural relationships between these various classes of graphs, which allows us, in particular, to enumerate all of them in their unlabeled forms.

Throughout this chapter, we denote by $\mathcal{BC}$ the species of bicolored graphs and by $\mathcal{BP}$ the species of bipartite graphs. The prefix $\mathcal{C}$ will indicate the connected analogue of such a species, so $\mathcal{CBP}$ is the species of connected bipartite graphs.

We are motivated by the graph-theoretic fact that each *connected* bipartite graph has exactly two bicolorings, and may be identified with an orbit of connected bicolored graphs under the action of $\mathfrak{S}_2$ where the nontrivial element τ reverses all vertex colors. We will hereafter treat all the various species of bicolored graphs as $\mathfrak{S}_2$-species with respect to this action and use the theory developed in Section 2.2 to pass to bipartite graphs.

3.2. Bicolored graphs

We begin our investigation by directly computing the $\mathfrak{S}_2$-cycle index for the species $\mathcal{BC}$ of bicolored graphs with the color-reversing $\mathfrak{S}_2$-action described previously. We will then use various methods from the species algebra of Section 2 to pass to other related species. To compute the $\mathfrak{S}_2$-cycle index $Z_{\mathcal{BC}}^{\mathfrak{S}_2}$ we compute separately $Z_{\mathcal{BC}}^{\mathfrak{S}_2}(e)$ and $Z_{\mathcal{BC}}^{\mathfrak{S}_2}(\tau)$.

3.2.1. Computing $Z_{\mathcal{BC}}^{\mathfrak{S}_2}(e)$

We construct the cycle index for the species $\mathcal{BC}$ of bicolored graphs in the classical way, which in light of our $\mathfrak{S}_2$-action will give $Z_{\mathcal{BC}}^{\mathfrak{S}_2}(e)$.

For each $n > 0$ and each permutation $\pi \in \mathfrak{S}_n$, we must count bicolored graphs on $[n]$ for which π is a color-preserving automorphism. To simplify some future calculations, we omit empty graphs and define $\mathcal{BC}[\varnothing] = \varnothing$. We note that the *number* of such graphs in fact depends only on the cycle type $\lambda \vdash n$ of the permutation π, so we can use the cycle index formula in equation (3) interpreted as a Γ-cycle index identity.

Fix some $n \geq 0$ and let $\lambda \vdash n$. We wish to count bicolored graphs for which a chosen permutation π of cycle type λ is a color-preserving automorphism. Each cycle of the permutation must correspond to a monochromatic subset of the vertices, so we may construct graphs by drawing bicolored edges into a given colored vertex set. If we draw some particular bicolored edge, we must also draw every other edge in its orbit under π if π is to be an automorphism of the graph. Moreover, every bicolored graph for which π is an automorphism may be constructed in this way Therefore, we direct our attention first to counting these edge orbits for a fixed coloring; we will then count colorings with respect to these results to get our total cycle index.

Consider an edge connecting two cycles of lengths m and n; the length of its orbit under the permutation is $\operatorname{lcm}(m,n)$, so the number of such orbits of edges between these two cycles is $mn/\operatorname{lcm}(m,n) = \gcd(m,n)$. For an example in the case $m = 4, n = 2$, see Fig. 2. The number of orbits for a fixed coloring is then $\sum \gcd(m,n)$ where the sum is over the multisets of all cycle lengths m of white cycles and n of black cycles in the permutation π. We may then construct any possible graph fixed by our permutation by making a choice of a subset of these cycles to fill with edges, so the total number of such graphs is $\prod 2^{\gcd(m,n)}$ for a fixed coloring.

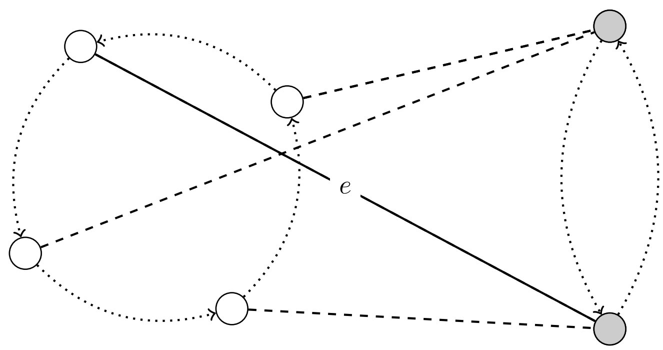

Figure 2: An edge e (solid) between two cycles of lengths 4 and 2 in a permutation and that edge's orbit (dashed)

We now turn our attention to the possible colorings of the graph which are compatible with a permutation of specified cycle type λ. We split our partition into two subpartitions, writing $\lambda = \mu \cup \nu$, where partitions are treated as multisets and $\cup$ is the multiset union, and μ represents the white cycles and ν the black. Then the total number of graphs fixed by such a permutation with a specified decomposition is

$$\operatorname{fix}(\mu, \nu) = \prod_{\substack{i \in \mu \\ j \in \nu}} 2^{\gcd(i,j)} \tag{11}$$

where the product is over the elements of μ and λ taken as multisets. Suppose that the multiplicities of the part i in the partitions λ, μ, and ν are l_i, m_i, and n_i, respectively. Then the l_i i-cycles of a permutation of cycle type λ can be colored so that m_i are white and n_i are black in $l_i!/(m_i!\, n_i!)$ ways. So in all there are $\prod_i l_i!/(m_i!\, n_i!) = z_\lambda/(z_\mu z_\nu)$ colorings associated with μ and ν, and

$$\operatorname{fix}(\lambda) = \frac{z_\lambda}{z_\mu z_\nu} \operatorname{fix}(\mu, \nu) = \sum_{\mu \cup \nu = \lambda} \frac{z_\lambda}{z_\mu z_\nu} \prod_{\substack{i \in \mu \\ j \in \nu}} 2^{\gcd(i,j)}.$$

Thus we have a formula for $Z_{\mathcal{BC}}^{\mathfrak{S}_2}(e)$:

Theorem 3.2.

$$Z_{\mathcal{BC}}^{\mathfrak{S}_2}(e) = \sum_{n > 0} \sum_{\substack{\mu, \nu \\ \mu \cup \nu \vdash n}} \frac{p_{\mu \cup \nu}}{z_\mu z_\nu} \prod_{i,j} 2^{\gcd(\mu_i, \nu_j)}. \tag{12}$$

Explicit formulas for the generating function for unlabeled bicolored graphs were obtained by Harary [7] using conventional Pólya-theoretic methods. Conceptually, this enumeration in fact largely mirrors our own. Harary uses the classical cycle index of the line group[2] of the complete bicolored graph of which any given bicolored graph is a spanning subgraph. He then enumerates orbits of edges under these groups using the Pólya enumeration theorem. This is clearly analogous to our procedure, which enumerates the orbits of edges under each specific permutation of vertices.

3.2.2. Calculating $Z^{\mathfrak{S}_2}_{\mathfrak{BC}}(\tau)$

Recall that the nontrivial element of $\tau \in \mathfrak{S}_2$ acts on bicolored graphs by reversing all colors.

We again consider the cycles in the vertex set $[n]$ induced by a permutation $\pi \in \mathfrak{S}_n$ and use the partition λ corresponding to the cycle type of π for bookkeeping. We then wish to count bicolored graphs on $[n]$ for which $\tau \cdot \pi$ is an automorphism, which is to say that π itself is a color-*reversing* automorphism. The number of bicolored graphs for which π is a color-reversing automorphism depends only on the cycle type λ. Each cycle of vertices must be color-alternating and hence of even length, so the partition λ must have only even parts. Once this condition is satisfied, edges may be drawn either within a single cycle or between two cycles, and as before if we draw in any edge we must draw in its entire orbit under π (since π is to be an automorphism of the underlying graph). Moreover, all graphs for which π is a color-reversing automorphism with a fixed coloring may be constructed in this way, so it suffices to count such edge orbits and then consider how colorings may be assigned.

We first determine the number of orbits of edges within a cycle of length $2n$; we hereafter describe such a cycle as having *semilength* n. There are exactly n^2 possible white-black edges in such a cycle. If n is even, then every edge lies in an orbit of size $2n$, so there are $n^2/(2n) = n/2$ orbits of edges. If n is odd, there are n edges joining diametrically opposed vertices, which have oppositive colors. These n edges are all in the same orbit. (See Fig. 3a for an illustration of these edges.) The remaining $n^2 - n$ edges are in orbits of size $2n$, so there are $(n^2 - n)/(2n) = (n - 1)/2$ of these orbits. (See Fig. 3b for an illustration of these edges.) Thus the total number of orbits for n odd is $(n + 1)/2$. In either case, the number of orbits is $\lceil n/2 \rceil$.

(a) A diameter d ($l = 3$)

(b) A non-diameter e ($l = 1$)

Figure 3: Both types of intra-cycle edges and their orbits on a typical color-alternating 6-cycle

Now consider an edge drawn between two cycles of semilengths m and n. The total number of possible white-black edges is $2mn$, each of which has an orbit length of $\operatorname{lcm}(2m, 2n) = 2\operatorname{lcm}(m, n)$. Hence, the total number of orbits is $2mn/(2\operatorname{lcm}(m, n)) = \gcd(m, n)$.

All together, then, the number of orbits for a fixed coloring of a permutation of cycle type 2λ (denoting the partition obtained by doubling every part of λ) is $\sum_i \lceil \lambda_i/2 \rceil + \sum_{i<j} \gcd(\lambda_i, \lambda_j)$. All valid bicolored graphs for a fixed coloring for which π is a color-preserving automorphism may be obtained uniquely by making some choice of a subset of this collection of orbits, just as in Section 3.2.1. Thus, the total number

[2]The *line group* of a graph is the group of permutations of edges induced by permutations of vertices.

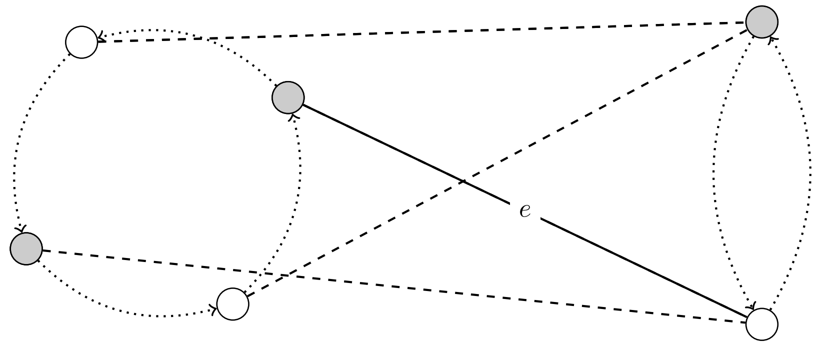

Figure 4: An edge e and its orbit between color-alternating cycles of semilengths 2 and 1

of possible graphs for a given vertex coloring is

$$\prod_i 2^{\lceil \lambda_i/2 \rceil} \prod_{i<j} 2^{\gcd(\lambda_i, \lambda_j)}, \tag{13}$$

which we note is independent of the choice of coloring. For a partition 2λ with $l(\lambda)$ cycles, there are then $2^{l(\lambda)}$ colorings compatible with our requirement that each cycle is color-alternating, which we multiply by (13) to obtain the total number of graphs for all permutations π with cycle type 2λ. Therefore we have a formula for $Z_{\mathcal{BC}}^{\mathfrak{S}_2}(\tau)$:

Theorem 3.3.

$$Z_{\mathcal{BC}}^{\mathfrak{S}_2}(\tau) = \sum_{\substack{n>0 \\ n \text{ even}}} \sum_{\lambda \vdash n/2} 2^{l(\lambda)} \frac{p_{2\lambda}}{z_{2\lambda}} \prod_i 2^{\lceil \lambda_i/2 \rceil} \prod_{i<j} 2^{\gcd(\lambda_i, \lambda_j)}. \tag{14}$$

3.3. Connected bicolored graphs

As noted in the introduction to this section, we may pass from bicolored to bipartite graphs by taking a quotient under the color-reversing action of $\mathfrak{S}_2$ only in the connected case. Thus, we must first pass from the species $\mathcal{BC}$ to the species $\mathcal{CBC}$ of connected bicolored graphs. It is a standard principle of graph enumeration that a graph may be decomposed uniquely into (and thus species-theoretically identified with) the set of its connected components. This same relationship holds in the case of bicolored graphs. Thus, the species $\mathcal{BC}$ of nonempty bicolored graphs is the composition of the species $\mathcal{CBC}$ of nonempty connected bicolored graphs into the species $\mathcal{E}^+ = \mathcal{E} - 1$ of nonempty sets:

$$\mathcal{BC} = \mathcal{E}^+ \circ \mathcal{CBC}. \tag{15}$$

Reversing the colors of a bicolored graph is done simply by reversing the colors of each of its connected components independently; thus, once we trivially extend the species $\mathcal{E}^+$ to an $\mathfrak{S}_2$-species by applying the trivial action, equation (15) holds as an identity of $\mathfrak{S}_2$-species for the color-reversing $\mathfrak{S}_2$-action described previously.

To use the decomposition in equation (15) to derive the $\mathfrak{S}_2$-cycle index for $\mathcal{CBC}$, we must invert the $\mathfrak{S}_2$-species composition into $\mathcal{E}^+$. We write $\Omega := (\mathcal{E}^+)^{\langle -1 \rangle}$ to denote the virtual species that is the inverse of $\mathcal{E}^+$ with respect to composition of species, following the notation of [12]. We can derive from [1, §2.5, equation (58c)] that its cycle index is

$$Z_\Omega = \sum_{k \geq 1} \frac{\mu(k)}{k} \log(1 + p_k) \tag{16}$$

where μ is the Möbius function. We can then rewrite equation (15) as

$$\mathcal{CBC} = \Omega \circ \mathcal{BC}.$$

From Theorem 2.11 we get a formula for the $\mathfrak{S}_2$-cycle index for $\mathcal{CBC}$:

Theorem 3.4.

$$Z^{\mathfrak{S}_2}_{\mathcal{CBC}} = Z_\Omega \circ Z^{\mathfrak{S}_2}_{\mathcal{BC}}. \tag{17}$$

Note that we could have avoided the use of virtual species by performing the inversion at the level of cycle indices.

3.4. Bipartite graphs

As we previously observed, connected bipartite graphs are naturally identified with orbits of connected bicolored graphs under the color-reversing action of $\mathfrak{S}_2$. Thus,

$$\mathcal{CBP} = \mathcal{CBC}/\mathfrak{S}_2.$$

By application of Theorem 2.13, we can then directly compute the cycle index of $\mathcal{CBP}$ in terms of previous results.

Theorem 3.5.

$$Z_{\mathcal{CBP}} = \overline{Z^{\mathfrak{S}_2}_{\mathcal{CBC}}} = \frac{1}{2}\left(Z^{\mathfrak{S}_2}_{\mathcal{CBC}}(e) + Z^{\mathfrak{S}_2}_{\mathcal{CBC}}(\tau)\right). \tag{18}$$

Since a bipartite graph is a set of connected bipartite graphs, we have $\mathcal{BP} = \mathcal{E} \circ \mathcal{CBP}$, and this gives the formula for the cycle index for bipartite graphs.

Theorem 3.6.

$$Z_{\mathcal{BP}} = Z_{\mathcal{E}} \circ Z_{\mathcal{CBP}}. \tag{19}$$

Theorem 3.6 allows us to compute the number of unlabeled bipartite graphs with n vertices. However, to do this calculation, we don't need the entire cycle index; we can do all of the calculations using ordinary generating functions. Specifically, let

$$\begin{aligned}
f_e(x) &= 1 + Z^{\mathfrak{S}_2}_{\mathcal{BC}}(e)(x, x^2, x^3, \dots) \\
f_\tau(x) &= 1 + Z^{\mathfrak{S}_2}_{\mathcal{BC}}(\tau)(x, x^2, x^3, \dots) \\
g_e(x) &= Z^{\mathfrak{S}_2}_{\mathcal{CBC}}(e)(x, x^2, x^3, \dots) \\
g_\tau(x) &= Z^{\mathfrak{S}_2}_{\mathcal{CBC}}(\tau)(x, x^2, x^3, \dots) \\
c(x) &= \tilde{Z}_{\mathcal{CBP}}(x) = Z_{\mathcal{CBP}}(x, x^2, x^3, \dots) \\
b(x) &= \tilde{Z}_{\mathcal{BP}}(x) = Z_{\mathcal{BP}}(x, x^2, x^3, \dots).
\end{aligned}$$

Then $c(x)$ is the ordinary generating function for connected bipartite graphs and $b(x)$ is the ordinary generating function for bipartite graphs. We have formulas for $f_e(x)$ and $f_\tau(x)$ as sums over partitions,

$$\begin{aligned}
f_e(x) &= \sum_{n=0}^{\infty} x^n \sum_{\substack{\mu,\nu \\ \mu\cup\nu\vdash n}} \frac{1}{z_\mu z_\nu} \prod_{i,j} 2^{\gcd(\mu_i,\nu_j)} \\
f_\tau(x) &= \sum_{n \text{ even}} x^n \sum_{\lambda\vdash n/2} \frac{2^{l(\lambda)}}{z_{2\lambda}} \prod_i 2^{\lceil \lambda_i/2\rceil} \prod_{i<j} 2^{\gcd(\lambda_i,\lambda_j)},
\end{aligned}$$

and $g_e(x)$ and $g_\tau(x)$ are related to $f_e(x)$ and $f_\tau(x)$ by

$$\begin{aligned}
f_e(x) &= \exp\left(\sum_{k=1}^{\infty} \frac{g_e(x^k)}{k}\right) \\
f_\tau(x) &= \exp\left(\sum_{k=0}^{\infty} \frac{g_\tau(x^{2k+1})}{2k+1} + \sum_{k=1}^{\infty} \frac{g_e(x^{2k})}{2k}\right),
\end{aligned}$$

which may be inverted to give

$$g_e(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log f_e(x^k)$$
$$g_\tau(x) = \sum_{k=0}^{\infty} \frac{\mu(2k+1)}{2k+1} \log f_\tau(x^{2k+1}) + \sum_{k=1}^{\infty} \frac{\mu(2k)}{2k} \log f_e(x^{2k}), \tag{20}$$

where μ is the Möbius function. Finally, $c(x) = \frac{1}{2}(g_e(x) + g_\tau(x))$ and $b(x) = \exp\left(\sum_{k=1}^{\infty} c(x^k)/k\right)$.

These calculations are essentially the same as Hanlon's [5], though he does not have our equation (20). Unlabeled bipartite graphs were first counted, using a different approach, by Harary and Prins [8].

In order to count bipartite blocks, which we accomplish in the next section, we do need the entire cycle index.

3.5. Nonseparable graphs

We now turn our attention to the notions of block decomposition and nonseparable graphs. A graph is said to be *nonseparable* if it is vertex-2-connected (that is, if there exists no vertex whose removal disconnects the graph); every connected graph then has a canonical "decomposition"[3] into maximal nonseparable subgraphs, often shortened to *blocks*. In the spirit of our previous notation, we we will denote by $\mathcal{NBP}$ the species of nonseparable bipartite graphs, our object of study.

The basic principles of block enumeration in terms of automorphisms and cycle indices of permutation groups were first identified and exploited by Robinson [13]. In [1, §4.2], a theory relating a species B of nonseparable graphs to the species C_B of connected graphs whose blocks are in B is developed using similar principles.

We extract two particular results, appearing as [1, equations 4.2.27 and 4.2.26a]. We note that the *derivative* F' of a species F [1, pp. 47–49] is defined by $F'[A] = F[A \cup \{*\}]$, where $*$ is not in A, and its cycle index satisfies $Z_{F'} = \partial Z_F / \partial p_1$. The *pointing* $F^\bullet$ of F [1, §2.1] is XF'.

Lemma 3.7. *Let B be a species of nonseparable graphs and let C denote the species of connected graphs whose blocks are in B. Then*

$$B = C\left(C^{\bullet\langle -1\rangle}\right) + XB' - X \tag{21a}$$

and

$$\mathcal{E}(B') = \frac{X}{C^{\bullet\langle -1\rangle}}. \tag{21b}$$

It is apparent that the class of nonseparable bipartite graphs is itself exactly the class of blocks that occur in block decompositions of connected bipartite graphs. Accordingly, we can apply Lemma 3.7 immediately:

Theorem 3.8.

$$\mathcal{NBP} = \mathcal{CBP}\left(\mathcal{CBP}^{\bullet\langle -1\rangle}\right) + X \cdot \mathcal{NBP}' - X, \tag{22a}$$

by equation (21a) *and*

$$\mathcal{NBP}' = \Omega\left(\frac{X}{\mathcal{CBP}^{\bullet\langle -1\rangle}}\right) \tag{22b}$$

by equation (21b).

We have already calculated the cycle index for the species $\mathcal{CBP}$, so the calculation of the cycle index of $\mathcal{NBP}$ is now simply a matter of algebraic expansion.

A generating function for labeled bipartite blocks was given by Harary and Robinson [9], where their analogue of equation (22) for the labeled exponential generating function for blocks comes from [4]. However, we could locate no corresponding unlabeled enumeration in the literature. The numbers of labeled and unlabeled nonseparable bipartite graphs for $n \leq 10$ as calculated using our method are given in Table 1.

[3] Note that this decomposition does not actually partition the vertices, since many blocks may share a single cut-point, a detail which significantly complicates but does not entirely preclude species-theoretic analysis.

A. Cycle indices of compositional inverse species

In Section 3.5, our results included two references to the compositional inverse $\mathcal{CBP}^{\bullet\langle -1\rangle}$ of the species $\mathcal{CBP}^{\bullet}$. Although we have not explored computational methods in depth here, the question of how to compute the cycle index of the compositional inverse of a specified species efficiently is worth some consideration. Several methods are available, including one developed in [1, 4.2.19] as part of the proof that arbitrary species have compositional inverses, but our preferred method is one of iterated substitution.

Suppose that Ψ is a species (with known cycle index) of the form $X + \Psi_2 + \Psi_3 + \dots$ where Ψ_i is the restriction of Ψ to structures on sets of cardinality i and that Φ is the compositional inverse of Ψ. Then $\Psi \circ \Phi = X$, so

$$X = \Psi \circ \Phi = \Phi + \Psi_2(\Phi) + \Psi_3(\Phi) + \dots$$

also. Thus

$$\Phi = X - \Psi_2(\Phi) - \Psi_3(\Phi) - \dots. \tag{23}$$

This recursive equation is the key to our computational method. To compute the cycle index of Φ to degree 2, we begin with the approximation $\Phi \approx X$ and then substitute it into the first two terms of equation (23): $\Phi \approx X - \Psi_2(X)$ and thus $Z_\Phi \approx Z_X - Z_{\Psi_2} \circ Z_X$. All terms of degree up to two in this approximation will be correct. To compute the cycle index of Φ to degree 3, we then take this new approximation $\Phi \approx X - \Psi_2(X)$ and substitute it into the first three terms of equation (23). This process can be iterated as many times as are needed; to determine all terms of degree up to n correctly, we need only iterate n times. With appropriate optimizations (in particular, truncations), this method can run very quickly on a personal computer to reasonably high degrees; we were able to compute $Z_{\mathcal{CBP}^{\bullet\langle -1\rangle}}$ to degree sixteen in thirteen seconds.

B. Numerical results

With the tools developed in Section 3, we can calculate the cycle indices of the species $\mathcal{NBP}$ of nonseparable bipartite graphs to any finite degree we choose using computational methods. This result can then be used to enumerate unlabeled bipartite blocks. We have done so here using Sage [16] and code listed in Appendix C. The resulting values appear in Table 1.

Table 1: Enumerative data for unlabeled bipartite blocks with n hedra

n	Unlabeled
1	1
2	1
3	0
4	1
5	1
6	5
7	8
8	42
9	146
10	956
11	6643
12	65921
13	818448
14	13442572
15	287665498
16	8099980771
17	300760170216
18	14791653463768
19	967055338887805
20	84368806391412395
21	9855854129239183783
22	1546801291978378704267
23	327092325302250220001201
24	93454432085788531687319514

C. Code listing

The functional equation (22) characterizes the cycle index of the species $\mathcal{NBP}$ of bipartite blocks. In this section we have used the the computer algebra system Sage [16] to adapt the theory into practical algorithms for computing the actual numbers of such structures. Python/Sage code to compute the coefficients of the ordinary generating function $\widetilde{\mathcal{NBP}}(x)$ of unlabeled bipartite blocks explicitly follows in listing 1.

Listing 1: Sage code to compute numbers of bipartite blocks (`bpblocks.sage`)

```
# Import needed code
from sage.combinat.species.stream import Stream, _integers_from #Infinite generator for lazy power series
from sage.combinat.species.generating_series import CycleIndexSeriesRing

# Set up helper variables
# We'll work with these cycle indices a lot
X = species.SingletonSpecies().cycle_index_series()
E = species.SetSpecies().cycle_index_series()

CIS = CycleIndexSeriesRing(QQ) #The ring of cycle index series with rational coefficients
p = SFAPower(QQ) #The ring of symmetric functions (power-sum basis) with rational coefficients

Z2 = SymmetricGroup(2) #The group of order 2
e = Z2.identity()
t = Z2.gen()

# Set up some helper functions
# Compute the derivative of a specified cycle index
def ci_deriv( f ):
    # Helper function to build a CIS corresponding to the $i$th term of the derivative of $f$
    def termbuilder( i ):
        # The first $i$ coefficients are $0$, the $(i+1)$th is the derivative of the
        # corresponding coefficient of $f$, and those afterwards are $0$ (filled
        # automatically by the CIS constructor).
        helperlist = [0]*i + [f.coefficient(i+1).derivative_with_respect_to_p1(), 0]
        # Now we let the CycleIndexSeriesRing constructor build a suitable CycleIndexSeries
        return CIS(helperlist)

    # The sum_generator constructor then builds a CycleIndexSeries which sums
    # all the terms we constructed above.
    return CIS.sum_generator(termbuilder(i) for i in _integers_from(0))

# Compute the pointing of a specified cycle index
ci_pointed = lambda f: ci_deriv(f) * X

# Compute the compositional inverse of a specified cycle index
# (Algorithm from Appendix A)
def ci_compinv( f ):
    # We must have that $\mathcal{F}_0 = \mathbf{0}$
    assert f.coefficient(0) == X.coefficient(0)
    # and that $\mathcal{F}_1 = \mathcal{X}$
    assert f.coefficient(1) == X.coefficient(1)
    # in order to apply this algorithm.

    # Following the result in Appendix A, if the conditions above are satisfied,
    # we have $\mathcal{F}^{\langle -1 \rangle} = \mathcal{X} - (\mathcal{F} - \mathcal{X}) \circ (\mathcal{F}^{\langle -1 \rangle})$.
    # The Sage cycle index define method handles this recursive definition.
    result = CIS()
    result.define(X - (f - X).compose(result))

    return result

# Compute the multiplicative inverse of a specified cycle index
# (Algorithm from [1, §2.5 exercise 7])
def ci_multinv( f ):
    # The result of [1] requires that $\mathcal{F}_0 \neq \mathbf{0}$.
    assert f.coefficient(0) != 0
```

```
    # If this condition holds, we have that $\mathcal{F}^{-1} = \sum_{i=0}^{\infty} (-\mathcal{F}_0)^i \cdot (F_+)^i$, just as for
    # ordinary formal power series.
    return CIS.sum_generator(f.coefficient(0)^(-i-1) * (f.coefficient(0) - f)^i for i in _integers_from(0))

# Compute a cycle index divided by $\mathcal{X}$, if possible
# (Helper for a later calculation where $\mathcal{F}^{-1}$ is not defined but $\frac{\mathcal{X}}{\mathcal{F}}$ is)
def ci_xdiv( f ):
    def p1_dropper( part ):
        assert 1 in part
        return p(part[:-1])

    def p1_dropper_sf( sf ):
        assert sf in SFAPower(QQ)
        return SFAPower(QQ)._apply_module_morphism(sf, p1_dropper)

    termbuilder = lambda i: CIS([0]*i + [p1_dropper_sf(f.coefficient(i+1)), 0])
    return CIS.sum_generator(termbuilder(i) for i in _integers_from(0))

# Compute the cycle index for the combinatorial logarithm $\Omega$
# (Algorithm from [12, Prop. 6]; somewhat faster applying the more general inversion algorithm above to compute $\mathcal{E}^{\langle -1 \rangle}$)

# First we define a helper function to compute the terms of $Z_{\Omega}$.
def omegaterm(n):
    assert n>=0
    assert n in ZZ

    if n == 0:
        return 0
    elif n == 1:
        return p[1]
    else:
        return 1/n * ((-1)^(n-1) * p[1]**n - sum(d * p([Integer(n/d)]).plethysm(omegaterm(d)) for d in divisors(n)[:-1]))

# Then we convert this to a Python generator for the benefit of the Sage
# CycleIndexSeries constructor.
def omegagen():
    for n in _integers_from(0):
        yield omegaterm(n)

# Then we feed this generator into our CycleIndexSeriesRing.
Omega = CIS(omegagen())

# Helper functions for working with partitions
# The union of two partitions is the partition corresponding to their multiset union
def union( mu, nu):
    return Partition(sorted(mu.to_list() + nu.to_list(), reverse=true))

# For a partition $\mu = [\mu_1, \mu_2, \ldots] \vdash m$ and a natural $n$, we define $n \cdot \mu := [n\mu_1, n\mu_2, \ldots] \vdash nm$.
# That is, $n \cdot \mu$ is the partition obtained from $\mu$ by multiplying the multiplicity of each part of $\mu$ by $n$.
def partmult( mu, n ):
    return Partition([part * n for part in mu.to_list()])

# Define the cycle index for $\mathcal{BC}$
# First we compute the number of graphs fixed under $e$ by permutations of cycle
# types $\mu$ and $\nu$ in accordance with equation (11).
def efixedbcgraphs( mu, nu ):
    return 2**(sum([gcd(i, j) for i in mu for j in nu]))

# Then we build a generator for the terms of the cycle index $Z_{\mathcal{BC}}^{\mathfrak{S}_2}(e)$.
def egen():
    yield p(0)
    for n in _integers_from(1):
```

```
        yield sum(p(union(pair[0], pair[1]))/(pair[0].aut() * pair[1].aut()) *
            efixedbcgraphs(pair[0], pair[1]) for pair in PartitionTuples(n, 2))

# Then we compute the number of graphs fixed under $\tau$ by a permutation of cycle type
# $\mu$ in accordance with equation (13).
def tfixedbcgraphs( mu ):
    return 2**(len(mu) + sum([integer_ceil(p/2) for p in mu]) + sum([gcd(mu[i], mu[j]) for
        i in range(0, len(mu)) for j in range(i+1, len(mu))]))

# Then we build a generator for the terms of the cycle index $Z^{\mathfrak{S}_2}_{\mathcal{BC}}(\tau)$.
def tgen():
    yield p(0)
    for n in _integers_from(1):
        yield p(0)
        yield sum(tfixedbcgraphs( mu ) * p(partmult(mu, 2))/partmult(mu, 2).aut() for mu
            in Partitions(n))

# Finally, we use the CycleIndexSeriesRing constructor (twice) to define $Z^{\mathfrak{S}_2}_{\mathcal{BC}}$ as a dictionary.
BC = {e: CIS(egen()), t: CIS(tgen())}

# Define the cycle index for $\mathcal{CBC}$
# The $\mathfrak{S}_2$-cycle index plethysm $Z^{\mathfrak{S}_2}_{\Omega} \circ Z^{\mathfrak{S}_2}_{\mathcal{BC}}$ requires
# the more complex calculation in Definition 2.10, which we compute termwise.
def CBCtermmap(term):
    if term == 0:
        return CIS(0)

    # We use the $\tau^p$ term stretched by $p$ for each part $p$ in our partition
    termbuilder = lambda part: prod(BC[t**p].stretch(p) for p in part)

    # We then return a CycleIndexSeries summing the new terms with their original coefficients
    return sum(coeff*termbuilder(part) for part,coeff in term)

# The $e$ term of $Z^{\mathfrak{S}_2}_{\mathcal{CBC}}$ is a classical plethysm.
# The $\tau$ term uses our more sophisticated calculation above.
CBC = {e: Omega.compose(BC[e]), t: CIS.sum_generator(CBCtermmap(Omega.coefficient(i)) for
    i in _integers_from(0))}

# Define the cycle indices for $\mathcal{CBP}$ and $\mathcal{BP}$ as in Theorem 3.5.
CBP = (CBC[e] + CBC[t]) * (1/2)

# Define the cycle index for $\mathcal{BP}$ as in Theorem 3.6.
BP = E.compose(CBP)

# Compute the cycle index for $\mathcal{NBP}$ as in Theorem 3.8.
CBP_pointed_compinv = ci_compinv(ci_pointed(CBP))
NBP = CBP.compose(CBP_pointed_compinv) + X *
    Omega.compose(ci_multinv(ci_xdiv(CBP_pointed_compinv))-1)

# Compute the first $k$ coefficients of $\widetilde{\mathcal{NBP}}(x)$
# (Note that this list is zero-indexed!)
NBP.isotype_generating_series().coefficients(k)
```

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